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BALANCING DOMAIN DECOMPOSITION FOR PROBLEMS WITH LARGE JUMPS IN COEFFICIENTS

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ABSTRACT. The Balancing Domain Decomposition algorithm uses in each iteration solution of local problems on the subdomains coupled with a coarse problem that is used to propagate the error globally and to guarantee that the possibly singular local problems are consistent. The abstract theory introduced recently by the first-named author is used to develop condition number bounds for conforming linear elements in two and three dimensions. The bounds are independent of arbitrary coefficient jumps between subdomains and of the number of subdomains, and grow only as the squared logarithm of the mesh size h. Computational experiments for two- and three-dimensional problems confirm the theory.

1. INTRODUCTION

Domain decomposition methods for solving elliptic boundary value problems have received much attention in the last few years. The main reason for the popularity of these methods is undoubtedly the need to take advantage of parallel computers, but many domain decomposition methods are efficient solvers in a classical uniprocessor environment as well. This paper is concerned with a domain decomposition preconditioner for linear, conforming finite elements for the elliptic equation $-\nabla \cdot (\sigma \nabla u) = f$ with the coefficients σ changing between the subdomains by orders of magnitude.

The main component of the domain decomposition algorithms of the type studied here is an approximate solver based on the solution of *local independent subproblems* on subdomains and a *global coarse problem* with one or a few unknowns for each subdomain to effect a global exchange of information between the subdomain solution. The composed approximate solver is then used as a preconditioner in the conjugate gradients method. It is well known that the absence of a coarse problem results in deterioration of convergence of the iterations with increasing number of subdomains [11, 14].

The Balancing Domain Decomposition (BDD) was introduced by Mandel [20] by adding a coarse problem to an earlier method of De Roeck and Le Tallec [11], known as the *Neumann-Neumann* method, based in turn on earlier work for the case of two subdomains [2] and on a closely related method of Glowinski and Wheeler for mixed

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problems [17]. The development of BDD was motivated by very good performance of the Neumann-Neumann preconditioner for real-world problems with strongly discontinuous coefficients for a small number of subdomains [11]. Algorithms similar to BDD but different in important aspects and convergence results also independent of coefficient jumps between subdomains were recently obtained by Dryja and Widlund [16], and by Sarkis [23]. For application of the BDD method to *mixed finite elements* and computational results on a parallel computer, see Cowsar, Mandel, and Wheeler [9]. For extensions of BDD to *plate bending* and performance results on a shared-memory parallel machine, see Le Tallec, Mandel, and Vidrascu [19].

In this paper, we extend the abstract theory from [20] by an elementary argument showing that the convergence of the BDD method is bounded independently on coefficient jumps of arbitrary size between subdomains. We obtain new bounds on convergence from the abstract theory by unifying the fundamental inequalities of the Domain Decomposition theory by Dryja and Widlund [12, 13, 26, 27] and Bramble, Pasciak, and Schatz [3, 5], and complementing them with some new results in the 2D case. In the 3D, we only need to use the inequalities from [5]. We show that the condition number after preconditioning is bounded by $C(1+\log^2(H/h))$, where H is the characteristic subdomain size and h is the characteristic element size, both in 2D and 3D. In the 3D case, such a bound was already given in [20] based on a different estimate from [11]. The theory is confirmed by computational experiments. Further numerical results, available in the technical report [21], demonstrate that the method behaves very well even in the case of general discontinuities of the coefficients and irregular subdomain shapes in 2D and in many cases in 3D as well.

The paper is organized as follows: §2 introduces the BDD algorithm. Abstract bounds on the condition number are given in §3, relying only on algebraic arguments. The assumptions of these bounds are verified for finite element discretizations in §4. Finally, §5 contains numerical results.

2. Formulation of the problem and algorithms

We will recall the notation and formulation of the algorithm, following [20]. Consider a system of linear algebraic equations

$$Ax = f, \tag{1}$$

with the $m \times m$, symmetric positive definite matrix A arising from a finite element discretization of a linear, elliptic, self-adjoint boundary value problem on a domain Ω. We assume the domain Ω to be split into nonoverlapping subdomains $\Omega_1, \ldots, \Omega_k$, each of which is the union of some of the elements. Let A_i be the local stiffness matrix corresponding to subdomain Ω_i, x_i be the vector of degrees of freedom corresponding to all elements in Ω_i, and let N_i denote the matrix with entries 0 or 1 mapping the degrees of freedom x_i into global degrees of freedom, i.e., $x_i = N_i^T x$. Then the stiffness matrix A is obtained by the standard subassembly process,

$$A = \sum_{i=1}^{k} N_i A_i N_i^T.$$

Each x_i is split into degrees of freedom $\bar{x}_i$ that correspond to $\partial\Omega_i$, called *interface degrees of freedom*, and the remaining *interior degrees of freedom* $\dot{x}_i$. The degrees of freedom on $\partial\Omega \cap \partial\Omega_i$ are assigned to the interiors. The subdomain stiffness matrices

and the 0-1 matrices N_i are then split accordingly:

$$x_i = \begin{pmatrix} \bar{x}_i \\ \dot{x}_i \end{pmatrix}, \qquad A_i = \begin{pmatrix} \bar{A}_i & B_i \\ B_i^T & \dot{A}_i \end{pmatrix}, \qquad N_i = (\bar{N}_i, \dot{N}_i).$$

Assume the subdomain matrices A_i to be symmetric and positive semidefinite and the submatrices $\dot{A}_i$ nonsingular. Without loss of generality, let the interface degrees of freedom be numbered first and the interior degrees of freedom second in the global numbering. Let

$$\Gamma = \bigcup_{i=1}^{k} \partial\Omega_i,$$

V_i be the space of the interface degrees of freedom for the subdomain Ω_i and V denote the space of all degrees of freedom on Γ, in a global numbering. After elimination of the interior degrees of freedom, the problem (1) reduces to a problem posed on the interface space V,

$$Su = g, \tag{2}$$

where S is the Schur complement

$$S = \sum_{i=1}^{k} \bar{N}_i S_i \bar{N}_i^T, \qquad S_i = \bar{A}_i - B_i \dot{A}_i^{-1} B_i^T \ . \tag{3}$$

The local Schur complements S_i are symmetric positive semidefinite and S is positive definite. Interpreting matrices as mappings, we have

$$S : V \to V, \qquad S_i : V_i \to V_i, \qquad \bar{N}_i : V_i \to V. \tag{4}$$

Throughout this paper, we denote $\langle u, v \rangle = u^T v$ and, for symmetric positive semidefinite B, $\langle u, v \rangle_B = \langle Bu, v \rangle$ and $\|u\|_B = (\langle u, u \rangle_B)^{1/2}$. The notation $u \perp v$ means $\langle u, v \rangle = 0$.

Much of the benefit of domain decomposition is obtained already by solving the reduced problem (2) by conjugate gradients with simple preconditioners such as an approximation to the diagonal of S, cf. [6, 7, 18]. Evaluation of the action of S_i can be implemented by solving a *Dirichlet* problem on Ω_i. The BDD method is based on the *Neumann-Neumann* preconditioner [11, 10], which requires the solution of a Neumann problem on every subdomain Ω_i (named so in contrast to the Neumann-Dirichlet preconditioner, which requires solving Neumann problems on some subdomains and uses the original Dirichlet problem on others).

An important design choice for the Neumann-Neumann preconditioner is the selection of *weight matrices* D_i that form a decomposition of unity on the interface space V,

$$\sum_{i=1}^{k} \bar{N}_i D_i \bar{N}_i^T = I. \tag{5}$$

A straightforward choice for D_i is a diagonal matrix with the diagonal elements being the reciprocal of the number of subdomains the degree of freedom is associated with. A better choice, which also guarantees a convergence bound independent of coefficient jumps between subdomains, is given in Theorem 3.3 below. For other possibilities, see [11] and §5 below.

The following algorithm defines a linear operator $M_{\text{N-N}}^{-1}$ for use as a preconditioner.

Algorithm 2.1 (Neumann-Neumann preconditioner, [11]). Given $r \in V$, compute $z = M_{\text{N-N}}^{-1} r$ as follows. Distribute r to subdomains,

$$r_i = D_i^T \bar{N}_i^T r,$$

solve the local problems

$$S_i u_i = r_i \tag{6}$$

on the subdomains, and average the results by

$$z = \sum_{i=1}^{k} \bar{N}_i D_i u_i. \quad \square$$

Since the matrices S_i are typically singular, De Roeck and Le Tallec [11] used a pseudoinverse obtained by replacing zero pivots in the Gaussian decomposition by positive values.

The BDD method adds a coarse problem as follows. Let $n_i = \dim V_i$, $0 \le m_i \le n_i$, and Z_i be $n_i \times m_i$ matrices of full column rank such that

$$\text{Ker } S_i \subset \text{ Range } Z_i, \qquad i = 1, \dots, k, \tag{7}$$

and let $W \subset V$ be defined by

$$W = \{v \in V \mid v = \sum_{i=1}^{k} \bar{N}_i D_i u_i, \ u_i \in \text{ Range } Z_i\}.$$

The space W will play the role of a coarse space just as in variational multigrid methods [22]. We say that $s \in V$ is *balanced* if

$$Z_i^T D_i^T \bar{N}_i^T s = 0, \qquad i = 1, \dots, k. \tag{8}$$

The process of replacing r by a balanced $s = r - Sw$, $w \in W$, will be called *balancing*. We are now ready to define the action $r \mapsto z = M^{-1} u$ of the BDD preconditioner.

Algorithm 2.2 (BDD preconditioner, [20]). Given $r \in V$, compute $M^{-1} r$ as follows. Balance the original residual by solving the auxiliary problem for unknown vectors $\lambda_i \in \mathbb{R}^{m_i}$,

$$Z_i^T D_i^T \bar{N}_i^T \big(r - S \sum_{j=1}^{k} \bar{N}_j D_j Z_j \lambda_j\big) = 0, \qquad i = 1, \dots, k, \tag{9}$$

and set

$$s = r - S \sum_{j=1}^{k} \bar{N}_j D_j Z_j \lambda_j, \qquad s_i = D_i^T \bar{N}_i^T s, \quad i = 1, \dots, k. \tag{10}$$

Find any solution u_i for each of the local problems

$$S_i u_i = s_i, \qquad i = 1, \dots, k, \tag{11}$$

balance the residual by solving the auxiliary problem for $\mu_i \in \mathbb{R}^{m_i}$,

$$Z_i^T D_i^T \bar{N}_i^T \big(r - S \sum_{j=1}^{k} \bar{N}_j D_j (u_j + Z_j \mu_j)\big) = 0, \qquad i = 1, \dots, k, \tag{12}$$

and average the result on the interfaces according to

$$z = \sum_{i=1}^{k} \bar{N}_i D_i (u_i + Z_i \mu_i). \quad \square \tag{13}$$

If some $m_i = 0$, then Z_i as well as the block unknowns μ_i and λ_i are void and the ith block equation is taken out of (9) and (12). The presence of the coarse problem now guarantees that the possibly singular local problems (11) are consistent, and the result of the algorithm does not depend on the choice of the solutions of (11), see [20].

In practice, the residual of the initial approximation should be balanced first as in (12); then the first balancing step (9) in every iteration can be omitted since the residual r received from the conjugate gradients algorithm is already balanced.

3. Algebraic theory

In this section, we give bounds on the condition number, relying on algebraic arguments only. These results apply to arbitrary linear systems of the form described in the preceding section. Their assumptions will be verified in the following section for systems obtained from a particular variant of the Finite Element Method.

The following theorem was proved in [20, Theorem 3.2] in the case when Range Z_i = Ker S_i, but the same proof applies here.

Theorem 3.1. *Algorithm 2.2 returns $z = M^{-1}r$, where M is symmetric positive definite and* cond $(M, S) = \lambda_{\max}(M^{-1}S)/\lambda_{\min}(M^{-1}S) \leq C$, *where*

$$C = \sup \left\{ \frac{\sum_{j=1}^{k} \| \bar{N}_j^T \sum_{i=1}^{k} \bar{N}_i D_i u_i \|_{S_j}^2}{\sum_{i=1}^{k} \|u_i\|_{S_i}^2} \mid u_i \in V_i,\ u_i \perp \text{Ker } S_i,\ S_i u_i \perp \text{Range } Z_i \right\}.$$

To motivate the bound given in Theorem 3.1, we need the concepts of glob and glob projection, defined as follows.

Definition 3.2. Any vertex, edge, and, in the 3D case, face, of Γ will be called a **glob**. A glob is understood to be relatively open; for example, an edge does not contain its endpoints. We will also identify a glob with the set of the degrees of freedom associated with it. The set of all globs will be denoted by $\mathcal{G}$. For a glob $G \in \mathcal{G}$, define the glob projection as follows: for a vector $u \in V$, $E_G u \in V$ is the vector that has the same values as u for all degrees of freedom in G, and all other degrees of freedom of $E_G u$ are zero. The glob projection in terms of the local degrees of freedom is $E_G^{ji} = \bar{N}_j^T E_G \bar{N}_i : V_i \to V_j$. $\square$

Note that any two distinct globs from $\mathcal{G}$ are disjoint, and $\Gamma = \bigcup_{i=1}^{k} \partial\Omega_i = \bigcup_{G \in \mathcal{G}} G$. The mappings E_G, E_G^{ij} correspond to zero-one matrices and satisfy

$$\sum_{G \in \mathcal{G}} E_G = I, \qquad \bar{N}_j^T \bar{N}_i = \sum_{G \in \mathcal{G}} E_G^{ji}, \qquad E_G^{ji} = E_G^{ji} E_G^{ii}, \tag{14}$$

and

$$G \subset \partial\Omega_i \cap \partial\Omega_j \iff E_G^{ji} \neq 0, \qquad G \subset \partial\Omega_i \iff E_G^{ii} \neq 0. \tag{15}$$

We are now ready for an abstract bound in the case when the matrices S_i are scaled by arbitrary positive numbers α_i, which corresponds to coefficient discontinuities of arbitrary size between the subdomains. The theorem is formulated and proved in terms of properties of matrices only.

Theorem 3.3. *Let $\alpha_i > 0$, $i = 1, \dots, k$, $t \geq 1/2$, and E_G^{ji}, $\bar{N}_i$, S_i, and Z_i satisfy* (3), (14), *and* (7). *Define D_i as the diagonal matrices*

$$D_i = \sum_{G: E_G^{ii} \neq 0} d(i,G) E_G^{ii}, \qquad d(i,G) = \frac{\alpha_i^t}{\sum_{j: E_G^{ji} \neq 0} \alpha_j^t}, \tag{16}$$

and assume that there exists a number R so that for all $i, j = 1, \dots, k$ and all G,

$$\frac{1}{\alpha_j} \|E_G^{ji} u_i\|_{S_j}^2 \leq \frac{1}{\alpha_i} R \|u_i\|_{S_i}^2 \tag{17}$$

for all u_i such that $u_i \perp \operatorname{Ker} S_i$, $S_i u_i \perp \operatorname{Range} Z_i$. Then the weight matrices D_i form a decomposition of unity (5), *and the preconditioner defined by Algorithm 2.2 satisfies*

$$\operatorname{cond}(M, S) \leq K^2 L^2 R, \tag{18}$$

where $K = \max_i |\{j \mid \bar{N}_j^T \bar{N}_i \neq 0\}|$, and $L = \max_{i,j} |\{G \mid E_G^{ji} \neq 0\}|$.

Proof. The property (5) follows from the definition (16) and from (14),

$$\sum_{i=1}^k \bar{N}_i^T D_i \bar{N}_i = \sum_{i=1}^k \sum_{G: E_G^{ii} \neq 0} d(i,G) E_G = \sum_{G \in \mathcal{G}} E_G = I \,.$$

Let j be fixed. Since there are at most K nonzero terms in the sum $\sum_{i=1}^k \bar{N}_j^T \bar{N}_i D_i u_i$, it follows by the triangle inequality and the Cauchy inequality that

$$\|\sum_{i=1}^k \bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j}^2 \leq \left(\sum_{i=1}^k \|\bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j} \right)^2 \leq K \sum_{i=1}^k \|\bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j}^2 \,,$$

and

$$\sum_{j=1}^k \|\sum_{i=1}^k \bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j}^2 \leq K^2 \sum_{i=1}^k \max_j \|\bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j}^2 \,. \tag{19}$$

If $E_G^{ji} \neq 0$, the coefficient $d(i,G)$ from (16) satisfies $d(i,G) \leq \alpha_i^t / (\alpha_i^t + \alpha_j^t)$, and it follows from (14) and from (17) that

$$\begin{aligned} \|\bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j} &\leq \sum_{G: E_G^{ji} \neq 0} \frac{\alpha_i^t}{\alpha_i^t + \alpha_j^t} \|E_G^{ji} u_i\|_{S_j} \leq \sum_{G: E_G^{ji} \neq 0} \frac{\alpha_i^{t-1/2} \alpha_j^{1/2}}{\alpha_i^t + \alpha_j^t} R^{1/2} \|u_i\|_{S_i} \\ &\leq L R^{1/2} \sup_{\rho > 0} \frac{\rho^{1/2}}{1 + \rho^t} \|u_i\|_{S_i} \leq L R^{1/2} \|u_i\|_{S_i} \,. \end{aligned}$$

Now by (19),

$$\sum_{j=1}^k \|\sum_{i=1}^k \bar{N}_j^T \bar{N}_i D_i u_i\|_{S_j}^2 \leq K^2 L^2 R \|u_i\|_{S_i}^2 \,,$$

which concludes the proof, owing to Theorem 3.1. □

Note that the constant K is the maximal number of adjacent subdomains Ω_j to any subdomain Ω_i plus one, and L is the maximal number of globs in any $\partial\Omega_i \cap \partial\Omega_j$. If $t > 1/2$, the estimate (18) can be slightly improved; in particular, if $t = 1$, analogously to the method of De Roeck and Le Tallec [11], one has $\operatorname{cond}(M, S) \leq K^2 L^2 R / 2$.

The related method of Dryja and Widlund [16] uses the coarse space W with $t = 1/2$ in (16), and the matrices S_i in (11) replaced by $S_i + c_i M_i$, M_i positive definite, to avoid solving singular problems. Sarkis [23] obtained an estimate for a similar method for nonconforming elements with any $t \geq 1/2$.

4. Theory for a finite element discretization

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2$ or $d = 3$) with a piecewise smooth boundary $\partial\Omega$, and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 disjoint, $|\Gamma_1| > 0$. Consider the model problem

$$Lu = f \text{ in } \Omega, \quad u = g \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2, \tag{20}$$

where

$$Lv = -\sum_{r,s=1}^{d} \frac{\partial}{\partial x_r}\left(\alpha(x)\beta_{rs}(x)\frac{\partial v(x)}{\partial x_s}\right), \tag{21}$$

with the coefficient matrix (β_{rs}) uniformly positive definite, bounded and piecewise smooth on Ω, and $\alpha(x)$ a positive constant in each subdomain Ω_i, i.e.,

$$\alpha(x) = \alpha_i > 0 \quad \text{for } x \in \Omega_i.$$

Let $\hat{\Omega}$ denote a reference domain of diameter $O(1)$ (e.g., square or cube in 2D or 3D, respectively) and assume that the subdomains Ω_i are of diameter $O(H)$ and shape regular, i.e.,

$$\Omega_i = F_i(\hat{\Omega}), \quad \|\partial F_i\| \leq CH, \quad \|\partial F_i^{-1}\| \leq CH^{-1}, \tag{22}$$

with ∂F_i the Jacobian and $\|\cdot\|$ the Euclidean $\mathbb{R}^d$ matrix norm.

Let $V_h(\Omega)$ be a conforming linear finite element space on a triangulation of Ω such that each subdomain Ω_i is the union of some of the elements, and the usual shape regularity and inverse assumption hold [8]. All functions $v \in V_h(\Omega)$ satisfy homogeneous boundary condition $u = 0$ on Γ_1.

Let $V_h(\Omega_i)$ be the space of the restrictions of functions in $V_h(\Omega)$ to Ω_i. In all the estimates below, C and c denote generic positive constants independent of the shape or size of Ω and Ω_i. Note that these constants may depend on the constant in (22) or on the regularity of the triangulation, but they are independent of h and H.

Following [4], [12] or [25], we define the scaled Sobolev norms

$$\|u\|^2_{1,\Omega_i} = |u|^2_{1,\Omega_i} + \frac{1}{H^2}|u|^2_{0,\Omega_i}, \qquad \|u\|^2_{1/2,\partial\Omega_i} = |u|^2_{1/2,\partial\Omega_i} + \frac{1}{H}|u|^2_{0,\partial\Omega_i},$$

where

$$|u|^2_{1,\Omega_i} = \int_{\Omega_i} |\nabla u(x)|^2 dx, \qquad |u|^2_{1/2,\partial\Omega_i} = \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|u(t) - u(s)|^2}{|t - s|^d} dt ds.$$

The advantage of this definition is that it allows us to restrict all of our considerations to the reference domain $\hat{\Omega}$ and use the mappings F_i to obtain the results for each Ω_i from the obvious norm equivalence

$$\begin{aligned} c\,\|u\|^2_{1,\Omega_i} &\leq \|u \circ F_i\|^2_{1,\hat{\Omega}} H^{d-2} \leq C\|u\|^2_{1,\Omega_i}, \\ c\,\|u\|^2_{1/2,\partial\Omega_i} &\leq \|u \circ F_i\|^2_{1/2,\partial\hat{\Omega}} H^{d-2} \leq C\|u\|^2_{1/2,\partial\Omega_i}. \end{aligned} \tag{23}$$

Assume that for each Ω_i, $\Gamma_1 \cap \partial\Omega_i$ is either empty or a part of $\partial\Omega_i$ of size bounded below by a fixed proportion of the size of $\partial\Omega_i$ so that the Poincaré inequality holds uniformly for all Ω_i and with the constant C independent of h and H,

$$|u|^2_{0,\Omega_i} \le CH|u|^2_{1,\Omega_i}, \qquad |u|^2_{0,\partial\Omega_i} \le CH^{1/2}|u|^2_{1/2,\partial\Omega_i} \tag{24}$$

for all $u \in V_h(\Omega_i)$ if $\Gamma_1 \cap \partial\Omega_i \neq \varnothing$ and for all $u \in V_h(\Omega_i)$, $\int_{\partial\Omega_i} u\,ds = 0$ if $\Gamma_1 \cap \partial\Omega_i = \varnothing$.

To apply Theorem 3.1, we first need to replace the S_i norm by the scaled $H^{1/2}$ norm. This is a standard result [3, 13, 26], which we state here for reference in a form suitable for our purposes. The scaling by α_i is obvious.

Lemma 4.1. *There exist constants $c > 0$, C independent of H or h so that*

$$c|u|^2_{1/2,\partial\Omega_i} \le \frac{1}{\alpha_i}\,\|u\|^2_{S_i} \le C|u|^2_{1/2,\partial\Omega_i}, \qquad \forall u \in V_h(\partial\Omega_i).$$

To derive the fundamental inequality (17) assumed in Theorem 3.3, we identify (by abuse of notation) V with $V_h(\Gamma)$ and V_i with $V_h(\partial\Omega_i)$. Then the glob projections are $E_G : V_h(\Gamma) \to V_h(\Gamma)$, and (17) becomes a bound on the increase of the $H^{1/2}$ norm when a function in $V_h(\partial\Omega_i)$ is changed by setting its values to zero on all nodes of $\partial\Omega_i \setminus G$.

We first consider the two-dimensional case, $\Omega \subset \mathbb{R}^2$. Since $\partial\Omega_i$ is one-dimensional, we may use the properties of the space $V_h(0, H)$ of piecewise linear functions on a uniform mesh with step h on the interval $[0, H]$. The following form of Discrete Sobolev Inequality was proved by Dryja [12].

Lemma 4.2. *There exists a constant C such that*

$$\|u\|^2_{L^\infty(0,H)} \le C\left(1 + \log\frac{H}{h}\right)\|u\|^2_{H^{1/2}(0,H)}, \qquad \forall u \in V_h(0, H).$$

We will also need the following bound for the $H^{1/2}$ norm of the extension by zero from an interval to the whole $\mathbb{R}$, proved by Bramble, Pasciak, and Schatz [3, Lemma 3.5].

Lemma 4.3. *There exists a constant C such that for all $u \in V_h(0, H)$ satisfying $u(0) = u(H) = 0$, $u = 0$ outside $(0, H)$,*

$$|u|^2_{1/2,\mathbb{R}} \le C\left(1 + \log\frac{H}{h}\right)\|u\|^2_{L^\infty(0,H)} + |u|^2_{1/2,(0,H)}.$$

An estimate of the $H^{1/2}$ norm of a "spike" function, obtained by sampling the value of a given function at one point, follows easily.

Lemma 4.4. *There exists a constant C such that for all $u \in V_h(0, H), 0 \le h \le 1$, and $v_0 \in V_h(\mathbb{R})$ defined by $v_0(0) = u(0)$, $v_0(x) = 0$ for $|x| \ge h$,*

$$|v_0|^2_{1/2,\mathbb{R}} \le C\left(1 + \log\frac{H}{h}\right)\|u\|^2_{1/2,(0,H)}.$$

Proof. Let $L = \|u\|_{L^\infty(0,H)}$. It follows from Lemma 4.3 that

$$|v_0|^2_{1/2,\mathbb{R}} \le C\left(1 + \log\frac{2h}{h}\right)\|v_0\|^2_{L^\infty(-h,h)} + |v_0|^2_{1/2,(-h,h)}. \tag{25}$$

Using linearity of v_0, we obtain

$$|v_0|^2_{1/2,(-h,h)} = \int_{-h}^{h}\int_{-h}^{h} \frac{|v_0(s)-v_0(t)|^2}{|s-t|^2} dsdt \le 4\,L^2, \tag{26}$$

because $\|v_0\|^2_{L^\infty(-h,h)} = |v_0(0)|^2 \le L^2$. Thus, $|v_0|^2_{1/2,(-h,h)} \le CL^2$. But $L^2 \le C(1+\log\frac{H}{h})\|u\|^2_{1/2,(0,H)}$ by Lemma 4.2, which concludes the proof. □

By subtracting such spikes at the endpoints, we can extend Lemma 4.3 to the case when the values of u at the endpoints are nonzero.

Lemma 4.5. *There exists a constant C so that for $u \in V_h(0,H)$ and $w \in V_h(\mathbb{R})$ such that $w = u$ on $[h, H-h]$, and $w(x) = 0$ for $x \le 0$, $x \ge H$,*

$$|w|^2_{1/2,\mathbb{R}} \le C\left(1+\log\frac{H}{h}\right)^2 \|u\|^2_{1/2,(0,H)}.$$

Proof. Define $u(x)$ to be zero for $x \in (-\infty,-h)\cup(H+h,\infty)$, and linear in $[-h,0]$ and $[H,H+h]$. Further, define v_0 and v_H by

$$v_0(x) = \begin{cases} u(0), & x = 0, \\ 0, & |x| \ge h, \end{cases}$$

v_0 linear in $[-h,0]$ and in $[0,h]$,

$$v_H(x) = \begin{cases} u(H), & x = H, \\ 0, & |x-H| \ge h, \end{cases}$$

v_H linear in $[H-h,H]$ and in $[H,H+h]$. Writing w as $w = u - v_0 - v_H$, and applying Lemma 4.3 and Lemma 4.4, we obtain

$$\begin{aligned}
|w|^2_{1/2,\mathbb{R}} &\le C\left(1+\log\frac{H}{h}\right)||w||^2_{L^\infty(0,H)} + |w|^2_{1/2,(0,H)} \\
&= C\left(1+\log\frac{H}{h}\right)||u||^2_{L^\infty(0,H)} + |w|^2_{1/2,(0,H)} \\
&\le C\left(1+\log\frac{H}{h}\right)||u||^2_{L^\infty(0,H)} + 3\big(|u|^2_{1/2,(0,H)} + |v_0|^2_{1/2,\mathbb{R}} + |v_H|^2_{1/2,\mathbb{R}}\big) \\
&\le C\left(\left(1+\log\frac{H}{h}\right)||u||^2_{L^\infty(0,H)} + |u|^2_{1/2,(0,H)} + (1+\log\frac{H}{h})||u||^2_{1/2,(0,H)}\right).
\end{aligned}$$

Application of Lemma 4.2 to $||u||_{L^\infty,(0,H)}$ concludes the proof. □

We are now ready for the estimate of the $H^{1/2}$ norm of the glob projections E_G, which shows that an arbitrary function in $V_h(\partial\Omega_i)$ can be decomposed into its glob parts with only a small increase in the $H^{1/2}$ norm.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. Then there exists a constant C not dependent of h or H, so that for any glob $G \in \mathcal{G}$ and for all $u \in V_h(\partial\Omega_i)$,*

$$\|E_G u\|^2_{1/2,\partial\Omega_i} \le C\left(1+\log\frac{H}{h}\right)^2 \|u\|^2_{1/2,G}.$$

Proof. In the 2D case, the proposition follows by using a mapping of $\partial\Omega_i$ onto an interval so that G maps to $(0,H)$, from Lemma 4.5 for G being an edge, and from Lemma 4.4 for G being a vertex.

In the 3D case, the proposition was proved for the case of G being a face of $\partial\Omega_i$ as Lemma 4.3 in [5]. In the case of G being an edge or a vertex of $\partial\Omega_i$, the proof follows from Lemma 4.2. and the proof of Lemma 4.1 of [5]. □

The bound on the condition number of the BDD algorithm follows.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, and the weight matrices D_i be diagonal with the entries given by* (16). *Then there exists a constant C independent of H, h and α_i, so that the condition number of the BDD method satisfies*

$$\text{cond}\,(M, S) \leq C\left(1 + \log^2 \frac{H}{h}\right).$$

Proof. We only need to verify the assumption (17) of Theorem 3.3. Lemma 4.1 allows to replace the S_i norms by the $H^{1/2}(\partial\Omega_i)$ seminorms, which may in turn be replaced by the $H^{1/2}(\partial\Omega_i)$ norms, owing to the Poincaré inequality (24). It remains to use Theorem 4.6. □

5. Computational results

The purpose of our computational tests was to demonstrate the fast convergence of the BDD method on complicated problems with varying coefficients.

In all of the following examples, the space V_h of the piecewise linear functions defined on a uniform rectangular mesh of stepsize h in 2D or 3D was used for the solution of the elliptic problem of the form (20),

$$-\text{div}(\sigma\nabla u) = 1 \text{ in } \Omega, \quad u = 1 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2,$$

with $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \varnothing$. The coefficient σ is an elementwise constant function, and k is the number of subdomains.

We have compared three algorithms: conjugate gradients applied to the reduced system (2) without preconditioning (denoted as CG in the tables), conjugate gradients with Neumann-Neumann preconditioner and the local singular problems (6) solved using the Moore-Penrose pseudoinverse (Algorithm 2.1, denoted as N-N), and conjugate gradients with the BDD preconditioner using Range Z_i = Ker S_i (Algorithm 2.2, denoted as BDD).

The stopping criterion for the iterations was based on

$$\frac{\lambda_{\max}(M^{-1}S)}{\lambda_{\min}(M^{-1}S)} \frac{\langle M^{-1}r, r\rangle}{\langle M^{-1}b, b\rangle} \leq \epsilon^2, \tag{27}$$

with r the current residual and b the right-hand side, which guarantees the relative precision of ϵ in the energy norm, cf. Ashby, Manteuffel, and Saylor [1]. The condition number $\lambda_{\max}(M^{-1}S)/\lambda_{\min}(M^{-1}S)$ reported in the tables and also used in (27) was estimated as the ratio of the extreme Ritz values for the Krylov space, computed from the eigenvalues of a tridiagonal matrix constructed from the Lanczos recursion in conjugate gradients. Number of iterations with * means that the criterion (27) was not satisfied when the maximum number of iterations was reached.

The 2D examples were computed by a prototype implementation of the BDD method programmed using the CLAM package [24]. In the two-dimensional test BDD implementation, the weights D_i were based on the diagonal entries of the Schur complements, as suggested in [11], because we had the diagonal entries of the

TABLE 1. 2D results for Poisson equation on unit square (Fig. 1, $\sigma_1 = \sigma_2 = 1$)

		CG		N-N		BDD	
h	k	iterations	cond.	iterations	cond.	iterations	cond.
1/20	4	31	63.426	10	45.592	6	1.231
1/40	16	65	338.008	38	3,190.710	13	2.004
1/50	25	82	555.515	58	8,691.200	13	2.046

TABLE 2. 2D checkerboard pattern (Fig. 1, $\sigma_1 = 10^3$, $\sigma_2 = 10^{-3}$)

				CG		N-N		BDD	
h	k	σ_1	σ_2	iter	cond.	iter	cond.	iter	cond.
1/30	9	10^3	10^{-3}	61	866.051	10	16.145	4	1.555
1/40	16	10^3	10^{-3}	130*	$6.89\ 10^7$	49	$5.61\ 10^6$	11	1.941
1/50	25	10^3	10^{-3}	116	1,571.66	25	63.939	7	1.629

TABLE 3. 2 x 2 checkerboard pattern for various σ_1, σ_2

				CG		N-N		BDD	
h	k	σ_1	σ_2	iter	cond.	iter	cond.	iter	cond.
1/40	4	10^1	10^{-1}	30*	220	10	268	7	1.22
1/40	4	10^2	10^{-2}	30*	436	9	2,574	5	1.04
1/20	4	10^4	10^{-4}	30*	506	7	10,280	4	1.00045

Schur complement available: Denote s^i_{ll} the diagonal entry of Schur complement S_i corresponding to global degree of freedom l. For subdomain Ω_i, the weight matrix D_i was constructed as diagonal with diagonal elements $d^i_{\psi_i(l)}$, where $\psi_i(l)$ is the local number in $\partial\Omega_i$ associated with the global degree of freedom l,

$$d^i_{\psi_i(l)} = \frac{s^i_{ll}}{\sum_{j:l\in\partial\Omega_j} s^j_{ll}},$$

which is essentially (16) computed node by node with the diagonal entries of S_i used instead of the scalars α_i. This choice of the weights was found to give good results [11]. The domain Ω was chosen to be the unit square and Γ_1 was the left-hand side of Ω.

The tests show that unlike for the CG and N-N method, the condition number and the number of iterations of the BDD method does not deteriorate for increasing number of subdomains (Table 1, Fig. 1), the coefficient σ varying by orders of magnitude between the subdomains (Table 2, Fig. 1), and increasing jumps in the coefficients (Table 3).

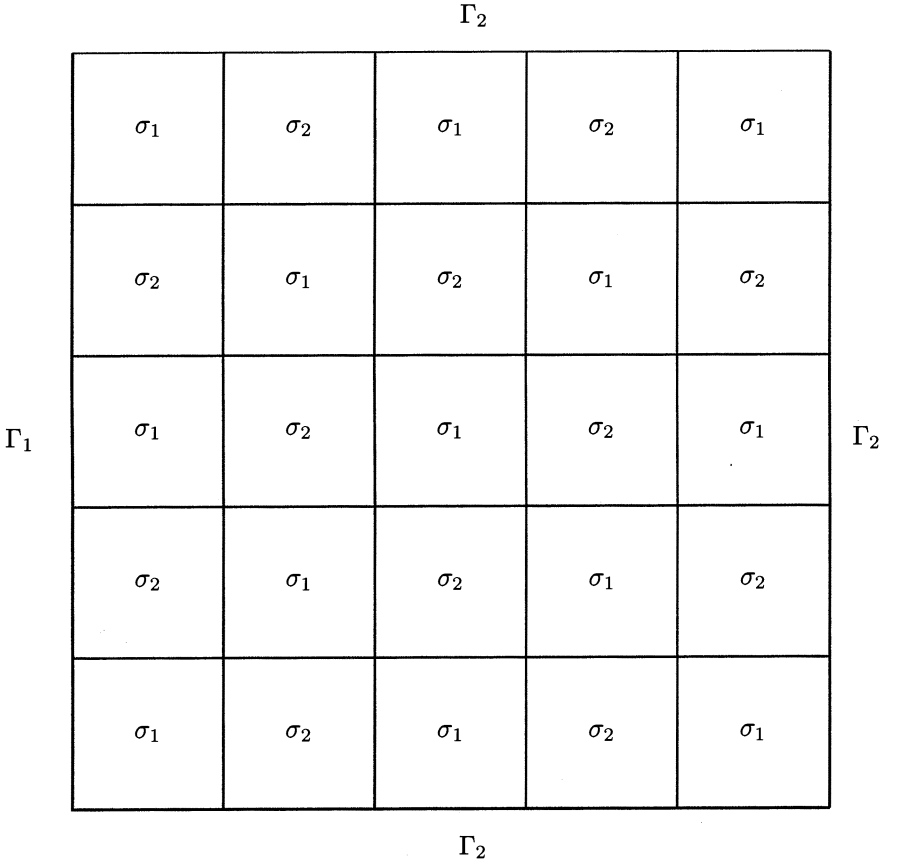

FIGURE 1. 2D checkerboard pattern

TABLE 4. 3D Poisson's equation, various h and number of subdomains

h_x	h_y	h_z	k	dof	iter	cond.
1/15	1/15	1/20	36	5376	25	3.5375
1/20	1/25	1/30	120	9246	37	4.6354
1/30	1/30	1/30	27	29791	22	4.8000

A FORTRAN implementation was used for the 3D experiments, with the action of S_i implemented in a straightforward way following the definition of the Schur complement (3). The implementation of the action of the inverse, that is, the solution of $S_i y = x$, relies on the obvious fact that y may equivalently be computed, using notation of §2, as solution of the sparse system

$$\begin{pmatrix} \bar{A}_i & B_i \\ B_i^T & \dot{A}_i \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

TABLE 5. 3D checkerboard pattern with alternating σ

h	k	σ_1	σ_2	dof	iter	cond.
1/25	125	10^0	10^0	17576	22	3.1154
1/25	125	10^1	10^{-1}	17576	19	2.4893
1/25	125	10^2	10^{-2}	17576	18	2.2071
1/25	125	10^3	10^{-3}	17576	16	2.0211
1/25	125	10^4	10^{-4}	17576	16	2.0023
1/25	125	10^5	10^{-5}	17576	16	2.0002
1/25	125	10^6	10^{-6}	17576	15	2.0000
1/25	125	10^7	10^{-7}	17576	15	2.0000

discarding z afterwards. Since the diagonal entries of S_i are not available, the weights were defined from $\alpha_i = \sigma_i$ by (16) with $t = 1$. The problem was set on unit cube Ω, with zero Dirichlet boundary condition on the whole $\partial\Omega$, and $\epsilon = 10^{-18}$ was used for the stopping criterion (27). Again, the results confirm the theory.

Finally, one should note that the l_2 norm of residual of the global solution was never larger than 20 times the l_2 residual of the reduced solution. For further numerical results, see [21].

FORTRAN 77 code that implements the method is available from `MGNET` by anonymous `ftp` to `casper.cs.yale.edu` in the directory `/mgnet/jmandel`. The code invokes user-supplied subroutines that implement the matrix-vector multiplications $S_i x_i$ and solution of the possibly singular systems $S_i z_i = r_i$.

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